Sub-exponential mixing rate for a class of Markov processes

S. A. Klokov

School of Mathematics, University of Leeds, UK (current address) & Omsk Branch of Institute of Mathematics, Omsk, Russia; e-mail: klokov@maths.leeds.ac.uk, klokov@iitam.omsk.net.ru

S. A. Klokov

Department of Mathematics, University of Kansas, Lawrence, KS, USA (current address) & School of Mathematics, University of Leeds, UK & Institute of Information Transmission Problems, Moscow, Russia; e-mail: veretennikov@math.ukans.edu, veretenn@iitp.ru

A. Yu. Veretennikov

Abstract

We establish sub-exponential bounds for the $\beta$-mixing rate and for the rate of convergence to invariant measures for discrete time Markov processes under recurrence conditions weaker than used for exponential inequalities and stronger than for polynomial ones.

Key words: Markov processes, recurrence, invariant measure, mixing coefficients, sub-exponential convergence.

1 Introduction

The goal of this paper is to establish sub-exponential mixing and convergence rate to equilibrium for a class of homogeneous Markov processes. What we

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2 The corresponding author is A. Yu. Veretennikov. Tel: +1 785 864 4452. Fax: +1 785 864 5255.
mean by \textit{sub-exponential} rate, is an upper bound of the type

\[ \beta_x(t) \leq C(x) e^{-ct}, \quad c, \delta, C(x) > 0, \quad t \geq 0. \quad (1) \]

Here \( \beta_x(t) \) is a \( \beta \)-mixing coefficient, see definition (6) below; certain other mixing coefficients or just distances between current and limiting distributions might stand in the left hand side of (1) instead, but we restrict ourselves to \( \beta \)-mixing coefficient and total variation norm. We do not use the term “sub-geometric” (which would be appropriate) only because the latter was recently used in several papers [39,12] with a different meaning: as polynomial or better than any polynomial convergence, which is still weaker than our notion of “sub-exponential” (1).

Assumptions will include two standard groups of conditions: recurrence and local mixing. Recurrence ones are formulated for the model (2) (see below) in terms of “drift and diffusion”, similar to continuous time setting, cf. [40,43,44] where drift and diffusion are the main and only components of the generator of the process and, hence, the most natural objects for assumptions. Other possible assumptions in terms of Lyapunov functions, as in the majority of papers are more general. However, they often leave an open question how to check them for a wide class of processes. Our point of view is that assumptions in terms of “drift and diffusion”, as for diffusion processes, are most natural.

Let us remind some history. Classical uniformly exponential convergence to equilibrium for Markov processes was established by Doeblin, Kolmogorov, Doob in the middle of the 20th century under various conditions, last version of which being known now as a “Doeblin type condition”, the name and final formulation belongs to Doob; see [11] for references. Starting from 70s, an increasing interest to non-uniform convergence for Markov processes arose, see [27,34], etc. An explanation of this interest is just because a lot of useful processes do not satisfy any uniform convergence bounds, while they do satisfy weaker properties such as a non-uniform convergence. It became clear that this non-uniform convergence relates to “local” Doeblin type condition and to hitting time bounds for some “petite sets”. The latter bounds were established in [19,27,34] et al. In the beginning of the theory of non-uniform bounds, only exponential convergence rate was treated, see [19,29,40,34], et al. A natural reason for studying exponential convergence bounds was probably that simple “Lyapunov functions” (see below) in corresponding cases can be established. In the last decade polynomial convergence bounds were investigated, [39,43,44,37,24,12,5] under essentially weaker assumptions. Same idea of Lyapunov functions was used in these works, too, although not that straightforward. In [39] a special sequence of auxiliary functions was constructed. To the authors knowledge, this was the first paper on the subject, i.e. on convergence bounds weaker than exponential ones, except “conditional” – that is, in terms other than processes generators – convergence and \( \beta \)-mixing polynomial.
bounds in [42,17]. In [43,44], for the same aim under more appropriate assumptions, “quasi-Lyapunov functions” were used, that is, functions imitating main properties of the Lyapunov ones (these are certain technical inequalities) up to a small controllable discrepancy. In [24,12] much simplified idea from [39] was used. Of course, in a broader sense all cited papers on non-uniform bounds explore extensions of the Lyapunov idea [31]. Notice that when we speak of “quasi-Lyapunov” functions, we do not mean that exact Lyapunov functions do not exist: e.g., for diffusions $E_x \tau$ ($\tau$ being some hitting time) is an example of such function. However, the question is how to construct explicitly a simple version of such or similar function which would allow to fulfill the calculus. For this aim, $E_x \tau$ cannot be regarded as an explicit formula. Hence, often an “approximate” or “quasi-Lyapunov” function is preferable in compare to an exact one.

We should also mention some other related results about hitting time polynomial inequalities, even though they do not state nor imply directly corresponding convergence nor mixing rate, [30,26,33,38,3]. First two papers in this list were pioneering, and both evidently were not appreciated at their time. It may be said that the use of both papers could accelerate a lot the investigation of polynomial non-uniform convergence (and mixing) bounds. E.g., “conditional” results from [42,17] combined with results and techniques from [30,26] would actually provide bounds similar to [43,44] and further in [24,12,13] long before the latter papers. Hitting time inequalities is the part of the technique used in [43,44], as well as in this paper. Though it is not possible to cite any related work, it is also worth to say that papers on Poisson equations “in the whole space” [16,36] use similar ideas; [36], in particular, uses directly polynomial bounds mentioned above.

Essentially, the approach from [43,44] was rewritten in [24] and [13] in a more abstract form using an assumption $PV \leq V - cV^\alpha + b1_C$, while the former approach was dealing with specific polynomial functions like $|x|^m$; the calculi are also close. Analogous methods based on [39] and same assumptions on $V$ were used in [12,25], too, to get polynomial convergence rate for the Metropolis–Hastings algorithm and for another related Markov chain model. We think our approach more clearly addresses the basic idea of how to get a specific polynomial or sub-exponential convergence or mixing rate.

In the meanwhile, there was a parallel process of exploration of so called mixing coefficients, a notion more general than convergence to equilibrium, very useful in limit theorems for “weakly dependent” random variables, cf. [22,14], et al. Most of works used various mixing coefficients assuming certain decreasing rate for them. A much smaller number of results were devoted to verifying different mixing conditions for stationary or Markov processes, see [20,21,6–8,1] et al. In the papers [40,42,17,43,44] both mixing and convergence rate were studied using the same approach based on the coupling method, see [2,35]. Its
source is usually (and correctly) attributed to Doeblin and Kolmogorov. Our results include inequalities for $\beta$-mixing coefficient. The latter was introduced in [46] under the name “complete regularity coefficient” for stationary processes, and now in a slightly extended form (for non-stationary processes) is known as Kolmogorov’s or $\beta$-coefficient, perhaps because both co-authors of [46] were students of the latter; later this coefficient was studied in [23] et al. This coefficient is an intermediate weak dependence characteristic between Rosenblatt’s $\alpha$-mixing and Ibragimov’s $\varphi$-mixing. Our point of view is that this coefficient and some its versions are very useful in situations when there is no $\varphi$-mixing (cf. [22,9]), which is fairly common.

The papers [43,44] were written in an attempt to understand polynomial convergence rate estimates suggested in [39], and apply them to mixing. Assumptions in [39] turned out to be non-optimal: in both [43] and [44], for certain classes of Markov processes, rather close polynomial bounds for convergence in total variation (as well as in similar norms with weight functions), and simultaneously for $\beta$-mixing were established under essentially less restrictive assumptions. Further, under original conditions used in [39], better “sub-exponential” bounds like (1) were proved in [32]. For some rather specific class of diffusions and under more restrictive assumptions sub-exponential convergence (as well as exponential and polynomial) was also established in [15].

In this paper we establish what we call “real” sub-exponential bounds for a class of Markov chains under recurrence conditions similar to those in [32]. The aim is to improve the bounds from [32] in the following sense: our power degree coefficient approaches 1 if recurrence assumptions “approach” assumptions implying exponential convergence rate (cf. [40,34]). This natural property fails for Malyshkin’s bounds. We also claim that analogous bounds with the same natural property hold true for diffusion processes under similar assumptions, too, though we do not consider this here.

Why sub-exponential bounds may be useful as such, not exponential nor polynomial? Nowadays there is a growing interest for using so called moderate deviations in various branches of stochastic analysis and its applications, see [4,28] et al. The use of Central Limit Theorem asymptotic is sometimes not sufficient to take into account important pre-limiting behavior features, while the powerful Large Deviation techniques which does allow to do it, is itself very vulnerable to perturbations, is not robust. Moderate Deviations provide some nice intermediate tool which combines certain advantages of both techniques while remaining much more robust to small errors than Large Deviations are. Sub-exponential mixing and convergence rates are very close to, and there is no doubt that they will be useful in the theory of moderate deviations, see [10,18].
2 The problem setting

Consider a homogeneous Markov process \((X_n, n \geq 0)\) in \(\mathbb{R}^d\) with the usual scalar product \((\cdot, \cdot)\), the norm \(|\cdot|\), and the family of Borel sets \(B(\mathbb{R}^d)\). The symbol \(\mathcal{F}_n^X\) stands for the \(\sigma\)-field generated by the sequence of random variables \((X_k), k \in I\). We write \(\mathcal{F}_n\) instead of \(\mathcal{F}_{\leq n}^X\) for the sake of simplicity. Denote by \(\mathbb{E}_x\) expectation of the process \((X_n)\) with the initial value \(X_0 = x\) and set \(\mathbb{P}_x(\cdot) = \mathbb{E}_x 1(\cdot), \mu_n = \mathcal{L}(X_n)\). We write \(\mathbb{E}_\mu\) and \(\mathbb{P}_\mu\), if \(\mathcal{L}(X_0) = \mu\). Symbols \(\land\) and \(\lor\) denote operations of taking minimum and maximum of real numbers respectively.

Consider the identity,

\[
X_{n+1} = \mathbb{E}\{X_{n+1}|\mathcal{F}_n\} + (X_{n+1} - \mathbb{E}\{X_{n+1}|\mathcal{F}_n\}).
\]

Let \(g\) be a Borel function satisfying \(g(X_n) = \mathbb{E}\{X_{n+1}|\mathcal{F}_n\}\) (a.s.). Denote \(V_{n+1} := X_{n+1} - \mathbb{E}\{X_{n+1}|\mathcal{F}_n\}\). We will use the representation of \(X_n\) in the form of non-linear autoregression,

\[
X_{n+1} = g(X_n) + V_{n+1}.
\] (2)

The definition of \(V_{n+1}\) implies that

\[
\mathbb{E}\{V_{n+1}|\mathcal{F}_n\} = 0.
\] (3)

In [44,43,40] the process \((X_n)\) was constructed directly by defining the function \(g(x)\) and introducing the i.i.d. sequence \((V_n)\).

Suppose that the following recurrence conditions are true:

- there exist positive constants \(R_0, C_0, r,\) and \(0 < p < 1\) such that

\[
|g(x)| \leq \begin{cases} C_0, & |x| \leq R_0, \\ |x|(1 - r/|x|^{1+p}), & |x| > R_0, \end{cases}
\] (4)

- there are \(K > 0\) and \(0 < \alpha \leq 1 - p\) such that

\[
\sup_n \mathbb{E}\{e^{k|V_{n+1}|^p}|\mathcal{F}_n\} < \infty, \quad 0 \leq k < K.
\] (5)

Condition (4) provides an “attraction” to the origin outside the ball \(\{x : |x| \leq R\}\). The parameter \(p\) regulates the force of the attraction. Two extremal cases \(p = 0\) and \(p = 1\) were regarded in [43] and [44] respectively; \(p = 0\) leads to an exponential mixing while \(p = 1\) provides a polynomial one. Condition (5)
requires finite sub-exponential moments of the “noise” \( (V_n) \) and implies existence of sub-exponential moments for \( X_n \). Additional assumptions on \( (X_n) \) will be formulated later.

The \( \beta \)-mixing coefficient is defined by the formula

\[
\beta_{n,x} = \sup_{m \geq 0} \mathbb{E}_x \text{var}_{B \in \mathcal{F}_{X_{n+m}}} \left( \mathbb{P}(B|\mathcal{F}_m) - \mathbb{P}(B) \right),
\]

(6)

where \( \text{var} \) denotes total variation of a signed measure \( \nu \). Also, the following version of this coefficient is often useful,

\[
\tilde{\beta}_n = \int \beta_{n,x} \mu_\infty(dx).
\]

(7)

Here \( \mu_\infty \) stands for the invariant measure. Assumptions of our theorems will guarantee that \( \mu_\infty \) exists and is unique.

Fix \( B \in \mathcal{B}(\mathbb{R}^d) \) and let \( \tau_0 = 0, \tau_{n+1} = \inf\{t > \tau_n : X_t \in B\} \). Define “the process on \( B \)”, \( X_n^B = X_{\tau_n} \). Denote by \( P^B(x,dy) \) the transition probability of the process \( (X_n^B) \).

Following [44], we say that the process \( (X_n) \) satisfies the local Doeblin condition, if for every \( R \) large enough and \( B = \{x \in \mathbb{R}^d : |x| \leq R\} \)

\[
\inf_{x,x' \in B} \int \min \left\{ \frac{P^B(x,dy)}{P^B(x',dy)}, 1 \right\} P^B(x',dy) = \kappa(R) > 0.
\]

(8)

Condition (8) provides non-singularity of the measures within the ball \( B \) and implies irreducibility. One might use a similar condition in terms of \( n \)-step transition probabilities.

3 Main result

We formulate our theorem in the section. All proofs are given in the end of the article.

**Theorem 1** Let \( (X_n) \) satisfy recurrence conditions (4)–(5), local Doeblin condition (8), and \( 0 < \delta < \alpha/(1+p) \). Then there exist \( c_0, c_1, C > 0 \), such that for any \( x = X_0 \) the following bounds for the rates of convergence of \( \mu_n = \mathcal{L}(X_n) \) to the invariant measure \( \mu_\infty \) and \( \beta \)-mixing hold true,

\[
\text{var} (\mu_n - \mu_\infty) \leq C e^{c_1|x|^\alpha} e^{-c_0 n^\delta},
\]

(9)
\[ \beta_{n,x} \leq C e^{c_1|x|^\alpha} e^{-c_0 n^\delta}, \]  
\[ \bar{\beta}_n \leq C e^{-c_0 n^\delta}. \]  

(10)  

(11)

4 Preliminaries

Let \( B_R = \{ x \in \mathbb{R}^d : |x| \leq R \} \). Suppose that our process \((X_n)\) starts from a point \( X_0 = x \) outside the ball \( B_R \), and set the stopping-time \( \tau = \inf \{ n \geq 1 : |X_n| \leq R \} \). Our nearest goal is to establish some inequalities for \((X_n)\), and to estimate sub-exponential moments of \( \tau \).

**Lemma 2** Let conditions (4), (5) hold true, \( 0 < \alpha < 1 - p \). Then for every \( 0 < r' < r \) the value of \( R_1 \geq R_0 \) can be chosen in such a way that for any \( R \geq R_1 \) and \( |x| > R \),

\[ \mathbb{E}_x \left( e^{k|X_{n+1}|^\alpha} - e^{k|X_n|^\alpha} \right) 1(A_n) \leq -k\alpha r' \mathbb{E}_x e^{k|X_n|^\alpha} |X_n|^{-1-p+\alpha} 1(A_n), \]  

(12)

where random events \( A_n = \{ |X_0| > R, |X_1| > R, \ldots, |X_n| > R \} \),

\[ \mathbb{E}_x e^{k|X_n|^\alpha} 1(n < \tau) \leq e^{k|x|^\alpha}, \]  

(13)

and there is \( C > 0 \) such that

\[ \mathbb{E}_x \tau \leq C e^{k|x|^\alpha}. \]  

(14)

If \( \alpha = 1 - p \) and \( r > k\alpha r_2 \), then (12) remains true for \( 0 < r' < r - k\alpha r_2 \), where \( r_2 = \sup_n \mathbb{E} \{ |V_{n+1}|^2 | F_n \} \).

Inequality (14) will be significantly improved in Lemma 3. However, it suffices to provide an existence of a unique invariant measure of a homogeneous irreducible Markov process (see Theorem 10.0.1 in [34]).

**Lemma 3** Let hypothesis of Lemma 2 hold true, \( 0 < \delta < \alpha/(1 + p) \), and \( R \) is chosen in accordance with Lemma 2. Then there are \( C > 0 \) and \( 0 < k < K \) such that

\[ \mathbb{E}_x e^{\tau^\delta} \leq C e^{k|x|^\alpha}. \]

Consider the direct product of two identical probability spaces where two independent copies of our Markov process \((X_n)\) and \((X'_n)\) with \( X_0 = x \) and \( X'_0 = x' \) are defined. Let \( \gamma = \inf \{ n \geq 1 : |X_n| \vee |X'_n| \leq \bar{R} \} \) is the Markov
moment of returning in the ball $B_{\bar{R}}$ for the “maximum” of $X_n$ and $X'_n$. We will study behavior of $|X_n| \vee |X'_n|$ and will establish some properties of $\gamma$ which are similar to corresponding ones of $\tau$.

**Lemma 4** Let conditions (4), (5) hold true, $0 < \alpha < 1 - p$. Then for every $0 < r'' < r$ the value of $\bar{R}_1 \geq R_0$ can be chosen in such a way that for any $\bar{R} \geq \bar{R}_1$ and any $|x| \vee |x'| > \bar{R}$,

$$
\mathbb{E}_{x,x'} \left( e^{k|X_{n+1}|^\alpha} + e^{k|X'_n|^\alpha} \right) 1(B_n) 
\leq \mathbb{E}_{x,x'} \left( e^{k|X_n|^\alpha} + e^{k|X'_n|^\alpha} \right) 1(B_n) 
- k\alpha r'' \mathbb{E}_{x,x'} e^{k(|X_n| \vee |X'_n|)^\alpha} \left( |X_n| \vee |X'_n| \right)^{-1-p+\alpha} 1(B_n), \quad (15)
$$

where random events

$$
B_n = \{|X_0| \vee |X'_0| > \bar{R}, |X_1| \vee |X'_1| > \bar{R}, \ldots, |X_n| \vee |X'_n| > \bar{R}\},
$$

and for every $n$

$$
\mathbb{E}_{x,x'} \left( e^{k|X_n|^\alpha} + e^{k|X'_n|^\alpha} \right) 1(n < \tau) \leq e^{k|x|^\alpha} + e^{k|x'|^\alpha}. \quad (16)
$$

If $\alpha = 1 - p$ and $r > k\alpha r_2$, then (15) remains true for $0 < r'' < r - k\alpha r_2$, where $r_2 = \sup_n \mathbb{E}\{|V_{n+1}|^2|\mathcal{F}_n\}$.

**Lemma 5** Let hypothesis of Lemma 4 hold true, $0 < \delta < \alpha/(1+p)$, and $\bar{R}$ is chosen in accordance with the same Lemma 4. Then there are $C > 0$ and $0 < k < K$ such that

$$
\mathbb{E}_{x,x'} e^{\gamma \delta} \leq C \left( e^{k|x|^\alpha} + e^{k|x'|^\alpha} \right).
$$

Assuming the local Doeblin condition holds true, we are able to obtain some results about sub-exponential moments with respect to the invariant measure $\mu_\infty$ and marginal distributions $\mu_n$.

**Lemma 6** Let hypothesis of Lemma 2 and the local Doeblin condition (8) hold true. Then for every $0 \leq k < K$,

$$
\mathbb{E}_{\mu_\infty} e^{k|X_n|^\alpha} = \int e^{k|x|^\alpha} \mu_\infty(dx) < \infty.
$$

**Lemma 7** Let hypothesis of Lemma 3 and the local Doeblin condition (8) hold true. Then for every $0 \leq k < K$,

$$
\sup_n \mathbb{E}_x e^{k|X_n|^\alpha} < \infty.
$$
5 Proofs

We use the scheme from [44] adapted for the sub-exponential case.

PROOF of Lemma 2 is divided into several steps for convenience.

0. We use some short notations to simplify formulas. Set \( X = X_n, V = V_{n+1}, \) \( g = g(X_n), \) then \( X_{n+1} = g + V. \)

1. Due to the recurrence condition (5), random variables from the sequence \( (V_n) \) have finite conditional expectations,

\[
\sup_n \mathbb{E} \{ e^{k|V_{n+1}|} | V_{n+1}^{|m} | \mathcal{F}_n \} < \infty, \tag{17}
\]

for every \( k \in [0, K) \) and \( m \geq 0. \) Also we use the following corollary of (17),

\[
\sup_n \mathbb{E} \{ e^{k|V_{n+1}|} | V_{n+1}^{|m} 1(|V_{n+1}| > N) | \mathcal{F}_n \} = o_N(1), \quad N \rightarrow \infty. \tag{18}
\]

2. We will estimate \( \mathbb{E}_x \left( e^{k|g+V|^\alpha} - e^{k|X|^\alpha} \right) 1(A_n) \). For an arbitrary \( \varepsilon \in (0, 1] \) split the expectation in two terms:

\[
I_1 = \mathbb{E}_x \left( e^{k|g+V|^\alpha} - e^{k|X|^\alpha} \right) 1(A_n \cap \{|V| > \varepsilon |X|^{1-\alpha}\}),
\]

\[
I_2 = \mathbb{E}_x \left( e^{k|g+V|^\alpha} - e^{k|X|^\alpha} \right) 1(A_n \cap \{|V| \leq \varepsilon |X|^{1-\alpha}\}).
\]

Each term will be estimated separately.

3. By (4) we have \( |g| \leq |X| \) on the set \( A_n \). Using inequalities \( (x+y)^\alpha \leq x^\alpha + y^\alpha \) for \( x, y \geq 0, 0 < \alpha \leq 1, \) and

\[
1(|V| > \varepsilon |X|^{1-\alpha}) \leq \frac{|V|^q}{\varepsilon^q |X|^q(1-\alpha)} 1(|V| > \varepsilon |X|^{1-\alpha}), \quad q > 0,
\]

one writes,

\[
I_1 \leq \mathbb{E}_x e^{k|X|^\alpha} e^{k|V|^\alpha} 1(A_n) 1(|V| > \varepsilon |X|^{1-\alpha}) \leq \varepsilon^{-q} \mathbb{E}_x e^{k|X|^\alpha} |X|^{-q(1-\alpha)} 1(A_n) \mathbb{E} \{ e^{k|V|^\alpha} |V|^q 1(|V| > \varepsilon R^{1-\alpha}) | \mathcal{F}_n \}.
\]

If we take \( q = (1 + p - \alpha)/(1 - \alpha) \) and use (18), then

\[
I_1 \leq o_R(1) \cdot \varepsilon^{-(1+p-\alpha)/(1-\alpha)} \mathbb{E}_x e^{k|X|^\alpha} |X|^{-1-p+\alpha} 1(A_n), \quad R \rightarrow \infty. \tag{19}
\]
4. We start to estimate $I_2$. Firstly, some preliminary boundaries are obtained. One writes,

$$
\left| \frac{g}{|X|} + \frac{V}{|X|} \right|^\alpha - 1 = \frac{\left( \frac{g}{|X|} + \frac{V}{|X|}, \frac{g}{|X|} + \frac{V}{|X|} \right)^{\alpha/2} - 1}{\left( 1 + 2 \left( \frac{g}{|X|}, \frac{V}{|X|} \right) + \frac{|g|^2 + |V|^2 - |X|^2}{|X|^2} \right)^{\alpha/2} - 1} \leq \alpha \left( \frac{g}{|X|}, \frac{V}{|X|} \right) + \frac{\alpha}{2} \frac{|g|^2 + |V|^2 - |X|^2}{|X|^2},
$$

since $(1 + z)^\alpha \leq 1 + \alpha z$ for $z \geq -1, 0 < \alpha \leq 1$.

Using previous calculations, we estimate $I_2$ as follows

$$
I_2 = \mathbb{E}_x e^{k|X|^\alpha} \left( e^{k|X|^{\alpha-1}} - 1 \right) 1(A_n \cap \{|V| \leq \varepsilon |X|^{1-\alpha}\}) \leq \mathbb{E}_x e^{k|X|^\alpha} \left( e^{k\alpha(J_1 + J_2)} - 1 \right) 1(A_n \cap \{|V| \leq \varepsilon |X|^{1-\alpha}\}),
$$

where $J_1 = \langle g/|X|, V/|X| \rangle$, $J_2 = (|g|^2 + |V|^2 - |X|^2)/(2|X|^{2-\alpha})$. We deal with $J_1$ and $J_2$ on the set $A_n \cap \{|V| \leq \varepsilon |X|^{1-\alpha}\}$, hence $|g| \leq |X|$ and

$$
|J_1| \leq \frac{|V|}{|X|^{1-\alpha}} \leq \varepsilon.
$$

Due to the recurrence condition (4),

$$
|g|^2 + |V|^2 - |X|^2 \leq |X|^2 \left( 1 - \frac{r}{|X|^{1-p}} \right)^2 + |V|^2 - |X|^2 \leq -2r|X|^{1-p} + r^2|X|^{-2p} + |V|^2,
$$

whence

$$
J_2 \leq -r|X|^{-1-p+\alpha} + \frac{r^2}{2} |X|^{-2-2p+\alpha} + \frac{|V|^2}{2} |X|^{-2+\alpha} =: J_3;
$$

moreover, the absolute value of the right hand side of (22) (we have denoted it as $J_3$) does not exceed

$$
rR^{-1-p+\alpha} + \frac{r^2}{2} R^{-2-2p+\alpha} + \frac{\varepsilon^2}{2} R^{-\alpha} = o_R(1), \quad R \to \infty.
$$

Thus, we are able to make $J_1 + J_3$ arbitrary small, choosing suitable $\varepsilon$ and $R$. 

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For every $\delta > 0$ there is a neighbourhood $U$ of 0 such that $e^x - 1 \leq x + (\frac{1}{2} + \delta)x^2$ for all $x \in U$, so from (21)–(23) we deduce that

\[
e^{k\alpha(J_1 + J_3)} - 1 \leq k\alpha(J_1 + J_3) + \left(\frac{1}{2} + \delta\right) k^2 \alpha^2 (J_1 + J_3)^2
\]
\[
\leq k\alpha(J_1 + J_3) + (1 + 2\delta) k^2 \alpha^2 (J_1^2 + J_3^2)
\]

on the set $A_n \cap \{|V| \leq \varepsilon|X|^{1-\alpha}\}$.

5. Since $\mathbb{E}\{V|F_n\} = 0$, using (17) and the trick from step 3 with the same $q = (1 + p - \alpha)/(1 - \alpha)$, we have

\[
\mathbb{E}_x e^{k|X|\alpha} J_1(A_n \cap \{|V| \leq \varepsilon|X|^{1-\alpha}\})
\]
\[
= \mathbb{E}_x e^{k|X|\alpha} \mathbf{1}_{A_n \cap \{|V| > \varepsilon|X|^{1-\alpha}\}}|F_n\}
\]
\[
- \mathbb{E}_x e^{k|X|\alpha} \mathbf{1}_{A_n \cap \{|V| > \varepsilon|X|^{1-\alpha}\}}|F_n\}
\]
\[
\leq \varepsilon^{-q} \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} \mathbf{1}_{A_n} \mathbb{E}\{\{|V| \leq \varepsilon|X|^{1-\alpha}\}|F_n\}
\]
\[
= o_R(1) \cdot \varepsilon^{-q} \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} \mathbf{1}_{A_n}, \quad R \to \infty.
\]

Further, from (21) one obtains,

\[
\mathbb{E}_x e^{k|X|\alpha} J_1^2(A_n \cap \{|V| \leq \varepsilon|X|^{1-\alpha}\})
\]
\[
\leq \mathbb{E}_x e^{k|X|\alpha} |X|^{-2+2\alpha} \mathbf{1}_{A_n} \mathbb{E}\{|V|^2|F_n\}
\]
\[
\leq \frac{r_2}{R^{1-p-\alpha}} \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} \mathbf{1}_{A_n}.
\]

If $\alpha < 1 - p$, then the fraction in the last term is $o_R(1)$, $R \to \infty$. If $\alpha = 1 - p$, then the value of the fraction is $r_2$.

6. Let us establish the bound for $\mathbb{E}_x e^{k|X|\alpha} J_3(A_n \cap \{|V| \leq \varepsilon|X|^{1-\alpha}\})$. Looking at (22), one can split the expectation into three terms:

\[
T_1 = -r \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} \mathbf{1}_{A_n \cap \{|V| \leq \varepsilon|X|^{1-\alpha}\}},
\]
\[
T_2 = \frac{r^2}{2} \mathbb{E}_x e^{k|X|\alpha} |X|^{-2+2\alpha} \mathbf{1}_{A_n \cap \{|V| \leq \varepsilon|X|^{1-\alpha}\}},
\]
\[
T_3 = \frac{1}{2} \mathbb{E}_x e^{k|X|\alpha} |X|^{-2+\alpha} |V|^2 \mathbf{1}_{A_n \cap \{|V| \leq \varepsilon|X|^{1-\alpha}\}}.
\]

One calculates,

\[
T_1 = -r \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} \mathbf{1}_{A_n}
\]
\[
+ r \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} \mathbf{1}_{A_n \cap \{|V| > \varepsilon|X|^{1-\alpha}\}},
\]
where the last summand does not exceed

\[ r\mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} 1(A_n) \mathbb{E}\{1(|V| > \varepsilon R^{1-\alpha})|\mathcal{F}_n\} \leq o_R(1) \cdot r\mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} 1(A_n). \]

Further,

\[ T_2 \leq \frac{r^2}{2R^{1+p}} \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} 1(A_n), \]

and

\[ T_3 \leq \frac{1}{2R^{1-p}} \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} 1(A_n) \mathbb{E}\{|V|^2|\mathcal{F}_n\} \leq \frac{M}{2R^{1-p}} \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} 1(A_n). \]

All the estimates on \( T_1, T_2 \) and \( T_3 \) imply

\[ \mathbb{E}_x e^{k|X|\alpha} J_3 1(A_n \cap \{|V| \leq \varepsilon |X|^{1-\alpha}\}) \leq (-r + o_R(1)) \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} 1(A_n), \quad R \to \infty. \tag{27} \]

Similarly, one shows that

\[ \mathbb{E}_x e^{k|X|\alpha} J_3 2(A_n \cap \{|V| \leq \varepsilon |X|^{1-\alpha}\}) = o_R(1) \cdot \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} 1(A_n). \]

7. Combining all the estimates (19), (24)–(28) together, in the case \( 0 < \alpha < 1 - p \) we get

\[ \mathbb{E}_x \left( e^{k|\sigma+V|\alpha} - e^{k|X|\alpha} \right) 1(A_n) \leq -k\alpha r' \mathbb{E}_x e^{k|X|\alpha} |X|^{-1-p+\alpha} 1(A_n), \]

where \( r' < r \) can be taken arbitrary close to \( r \), if we choose \( \varepsilon \) small and \( R \) large enough. For the case \( \alpha = 1 - p \) the constant \( r' < r - (1 + 2\delta)k\alpha r_2 \) and can be taken arbitrary close to \( r - k\alpha r_2 \). Inequality (12) is proved.

8. Since \( A_n \subset A_{n-1} \) we obtain (13) from (12) by iterations,

\[ \mathbb{E}_x e^{k|X_n|\alpha} 1(A_n) \leq \mathbb{E}_x e^{k|X_{n-1}|\alpha} 1(A_{n-1}) \leq \ldots \leq \mathbb{E}_x e^{k|X_0|\alpha} 1(A_0) = e^{k|x|\alpha}. \]
9. Let $c = \inf_{x \geq R} e^{k|\alpha|^\alpha |x|^{-1-p+\alpha}} > 0$. Then $e^{k|\alpha|^\alpha |X_n|^{-1-p+\alpha}} \geq c$ on every set $A_n$. From (12) we deduce

$$r'c\mathbb{E}_x 1(A_n) \leq \mathbb{E}_x e^{k|\alpha|^\alpha} 1(A_n) - \mathbb{E}_x e^{k|\alpha|^\alpha} 1(A_{n+1}).$$

Hence, the inequality (14) holds with $C = 1/(cr')$, summing over $n$. \qed

Remark 8 Estimates (26) and (27) show that in the case $\alpha > 1 - p$ our method does not work. It seems that for small values of $p$ an “attraction force” to the origin is not enough to provide recurrence in terms of sub-exponential moments with a degree $\alpha > 1 - p$, at least if we use this method of proof.

**PROOF of Lemma 3.** Using notations of Lemma 2, one writes the Abel transformation, or summation by parts,

$$\mathbb{E}_x e^{n\delta} = \sum_{n=0}^{\infty} e^{n\delta} \mathbb{E}_x 1(\tau = n)$$

$$\leq 1 + \sum_{n=0}^{\infty} \left(e^{(n+1)\delta} - e^{n\delta}\right) \mathbb{E}_x 1(A_n)$$

$$\leq 1 + (e-1) \sum_{n=0}^{\infty} e^{n\delta} \mathbb{E}_x 1(A_n).$$

We will estimate $e^{n\delta} \mathbb{E}_x 1(A_n)$ in order to provide the convergence of the series. By Lemma 2, the inequality (12) is true for some $k < K$ and $r' > 0$. Multiplying this inequality by $e^{n\delta}$ and using equivalent transformations, we rewrite it as

$$\mathbb{E}_x e^{n\delta} e^{k|\alpha|^\alpha} 1(A_n) - \mathbb{E}_x e^{(n-1)\delta} e^{k|\alpha|^\alpha} 1(A_n)$$

$$\leq -r'\mathbb{E}_x e^{n\delta} e^{k|\alpha|^\alpha} |X_n|^{-1-p+\alpha} 1(A_n)$$

$$+ \mathbb{E}_x \left(e^{n\delta} - e^{(n-1)\delta}\right) e^{k|\alpha|^\alpha} 1(A_n).$$

Since $e^{n\delta} - e^{(n-1)\delta} \sim \delta n^{-1+\delta} e^{n\delta}$, $n \to \infty$, we have $e^{n\delta} - e^{(n-1)\delta} \leq 2\delta n^{-1+\delta} e^{n\delta}$ for large $n$. One obtains,

$$\mathbb{E}_x e^{n\delta} e^{k|\alpha|^\alpha} 1(A_n) - \mathbb{E}_x e^{(n-1)\delta} e^{k|\alpha|^\alpha} 1(A_n)$$

$$\leq -\mathbb{E}_x e^{n\delta} e^{k|\alpha|^\alpha} \left(\frac{r'}{|X_n|^{1+p-\alpha}} - \frac{2\delta}{n^{1-\delta}}\right) 1(A_n). \tag{28}$$
Let \( c := \inf_{x \geq R} e^{k|x|^\alpha} |x|^{-1-p+\alpha} \); notice that \( c > 0 \). Consider the set,

\[
D = \left\{ r' \geq \frac{4\delta}{n^{1-\delta}} \right\}
\]

and an identity \( 1 = 1(D) + 1(D^c) \). The right hand side of (28) multiplied by \( 1(D) \) is less than or equal to

\[
-\frac{r'}{2} \mathbb{E}_x e^{\alpha \delta} e^{k|X_n|^\alpha} |X_n|^{-1-p+\alpha} 1(A_n) 1(D)
\]

\[
\leq -\frac{Cr'}{2} \mathbb{E}_x e^{\alpha \delta} 1(A_n) + \frac{r'}{2} \mathbb{E}_x e^{\alpha \delta} e^{k|X_n|^\alpha} |X_n|^{-1-p+\alpha} 1(A_n) 1(D^c)
\]

\[
\leq -\frac{Cr'}{2} \mathbb{E}_x e^{\alpha \delta} 1(A_n) + 2\delta \mathbb{E}_x e^{\alpha \delta} e^{k|X_n|^\alpha} n^{-1+\delta} 1(A_n) 1(D^c).
\]

On the set \( D^c \) the right hand side of (28) is less than or equal to

\[
2\delta \mathbb{E}_x e^{\alpha \delta} e^{k|X_n|^\alpha} n^{-1+\delta} 1(A_n) 1(D^c).
\]

Summing the last two estimates (and dropping \( n^{-1+\delta} \leq 1 \) in some terms), we deduce from (28) that

\[
\frac{Cr'}{2} \mathbb{E}_x e^{\alpha \delta} 1(A_n) \leq \mathbb{E}_x e^{(n-1)\delta} e^{k|X_n|^\alpha} 1(A_n) - \mathbb{E}_x e^{\alpha \delta} e^{k|X_{n+1}|^\alpha} 1(A_{n+1})
\]

\[
+ 4\delta \mathbb{E}_x e^{\alpha \delta} e^{k|X_n|^\alpha} 1(A_n) 1(D^c). \tag{29}
\]

For any \( \varepsilon > 0 \) and \( q > 0 \) the following inequality is true,

\[
1(a < b) \leq e^{-\varepsilon a^q + \varepsilon b^q}, \quad a, b > 0,
\]

whence

\[
1(D^c) = 1 \left( r' n^{1-\delta} < 4\delta |X_n|^{1+p-\alpha} \right) \leq e^{-\varepsilon (r' n^{1-\delta})^q + \varepsilon (4\delta |X_n|^{1+p-\alpha})^q},
\]

and the last term in (29) does not exceed

\[
4\delta \mathbb{E}_x e^{\alpha \delta - \varepsilon (r')^q n^{(1-\delta)} - (4\delta)^q |X_n|^{(1+p-\alpha)} q} 1(A_n).
\]

Let \( q = \alpha/(1+p-\alpha) \), then the second exponent has a degree \((k+\varepsilon(4\delta)^q)|X_n|^\alpha\). Taking \( \varepsilon > 0 \) small enough, we get \( k + \varepsilon(4\delta)^q =: k' < K \).
If $\delta < \alpha/(1+p)$, then $q(1-\delta) > \delta$, so
\[
e^{-n^\delta - c(r')qn\delta(1-\delta)} \leq e^{-n^\delta}
\]
for all $n$ large enough.

Gathering all the estimates together, one gets,
\[
\frac{c r'}{2} E_x e^{n^\delta} 1(A_n) \leq E_x e^{(n-1)^\delta} e^{k|X_n|^\alpha} 1(A_n) - E_x e^{n^\delta} e^{k|X_{n+1}|^\alpha} 1(A_{n+1}) \\
+ 4\delta e^{-n^\delta} E_x e^{k'|X_n|^\alpha} 1(A_n).
\]
The series $\sum_{n=1}^\infty e^{-n^\delta}$ converges, and by Lemma 2, $E_x e^{k'|X_n|^\alpha} 1(n < \tau) \leq e^{k'|x|^\alpha}$. Thus, the required inequality (with $k'$ instead of $k$) follows by summing over $n$. □

**PROOF of Lemma 4** is based on the lines of Lemma 2. Let $R$ be so large that Lemma 2 is applicable with such $r'$ that $r'' < r' < r$. The value of $\tilde{R} > R$ will be chosen later. One writes,
\[
1(B_n) = 1(B_n \cap \{|X_n| > R, |X'_n| > R\}) \\
+ 1(B_n \cap \{|X_n| \leq R, |X'_n| > R\}) \\
+ 1(B_n \cap \{|X_n| > R, |X'_n| \leq R\}),
\]
and one represents $E_{x,x'} (e^{k|X_{n+1}|^\alpha} + e^{k'|X_{n+1}|^\alpha}) 1(B_n)$ as a sum of three terms, $T_1$, $T_2$ and $T_3$ respectively.

Since $(X_n)$ and $(X'_n)$ are independent and inclusions
\[
B_n \cap \{|X_n| > R, |X'_n| > R\} \subset \{|X_n| > R\}, \\
B_n \cap \{|X_n| > R, |X'_n| > R\} \subset \{|X'_n| > R\}
\]
hold true, then the calculus in the proof of Lemma 2 which provides (12) can be used. One estimates,
\[
T_1 \leq E_{x,x'} \left( e^{k|X_n|^\alpha} + e^{k'|X'_n|^\alpha} \right) 1(B_n \cap \{|X_n| > R, |X'_n| > R\}) \\
- r'E_{x,x'} \left( e^{k|X_n|^\alpha} |X_n|^{1-p+\alpha} + e^{k'|X'_n|^\alpha} |X'_n|^{1-p+\alpha} \right) \\
\times 1(B_n \cap \{|X_n| > R, |X'_n| > R\}) \\
\leq E_{x,x'} (e^{k|X_n|^\alpha} + e^{k'|X'_n|^\alpha}) 1(B_n \cap \{|X_n| > R, |X'_n| > R\})
\]
\( - r' \mathbb{E}_{x,x'} e^{k(|X_n| + |X_n'|)}^\alpha (|X_n| \lor |X_n'|)^{-1-p+\alpha} \times 1(B_n \cap \{|X_n| > R, |X'_n| > R\}) \)

with the same \( r' \) as in Lemma 2.

In the case of \( T_2 \) we may write,

\[
\mathbb{E}_{x,x'} e^{k|X_n+1|^{\alpha}} 1(B_n \cap \{|X_n| \leq R, |X'_n| > R\}) \\
\leq \mathbb{E}_{x,x'} e^{k|X_n|^\alpha + k|X_{n+1}|^\alpha} 1(B_n \cap \{|X_n| \leq R, |X'_n| > R\}) \\
\leq e^{kR \alpha} \mathbb{E}_{x,x'} 1(B_n \cap \{|X_n| \leq R, |X'_n| > R\}) \mathbb{E}\{e^{k|V_{n+1}|^{\alpha}} |\mathcal{F}_n\} \\
\leq C_R \mathbb{E}_{x,x'} 1(B_n \cap \{|X_n| \leq R, |X'_n| > R\}),
\]

due to (17). Since \( |X'_n| > \bar{R} \) on the set \( B_n \), let us take \( \bar{R} \) so large that

\[
(r' - r'' e^{k|X'_n|^\alpha} |X'_n|^{-1-p+\alpha}) > C_R.
\]

Then

\[
T_2 \leq \mathbb{E}_{x,x'} e^{k|X'_n|^{\alpha}} 1(B_n \cap \{|X_n| \leq R, |X'_n| > R\}) \\
- r'' \mathbb{E}_{x,x'} e^{k|X'_n|^\alpha} |X'_n|^{-1-p+\alpha} 1(B_n \cap \{|X_n| \leq R, |X'_n| > R\}) \\
\leq \mathbb{E}_{x,x'} \left( e^{k|X_n|^\alpha} + e^{k|X'_n|^\alpha} \right) 1(B_n \cap \{|X_n| \leq R, |X'_n| > R\}) \\
- r'' \mathbb{E}_{x,x'} e^{k(|X_n| \lor |X'_n|)^\alpha} (|X_n| \lor |X'_n|)^{-1-p+\alpha} \\
\times 1(B_n \cap \{|X_n| \leq R, |X'_n| > R\}).
\]

The calculus for \( T_3 \) is similar. Summing up the estimates for \( T_1, T_2 \) and \( T_3 \), one gets (15). This relation (15) is quite similar to (12), so that the last step, obtaining (16), is the same as the corresponding one in Lemma 2. \( \square \)

**PROOF of Lemma 5** follows the lines of the proof of Lemma 3 with some modifications. It is sufficient to show that the series \( \sum_{n=0}^{\infty} e^{n\delta} \mathbb{E}_{x,x'} 1(B_n) \) converges. Multiplying (15) by \( e^{n\delta} \), we proceed like in Lemma 3 and obtain instead of (28) the following inequality,

\[
\mathbb{E}_{x,x'} e^{n\delta (e^{k|X_{n+1}|^{\alpha}} + e^{k|X'_n+1|^{\alpha}})} 1(B_n) \\
- \mathbb{E}_{x,x'} e^{(n-1)\delta (e^{k|X_n|^\alpha} + e^{k|X'_n|^\alpha})} 1(B_n) \\
\leq - \mathbb{E}_{x,x'} e^{n\delta} e^{k(|X_n| \lor |X'_n|)^\alpha} \frac{r''}{(|X_n| \lor |X'_n|)^{1+p-\alpha} - \frac{4\delta}{n^{1-\delta}}} 1(B_n).
\]
Denote \( \tilde{c} := \inf_{|x| \geq R} e^{k|x|^\alpha} |x|^{-1-p+\alpha} > 0 \), and consider the set

\[
\tilde{D} = \left\{ \frac{r''}{(|X_n| \vee |X'_n|)^{1+p-\alpha}} \geq \frac{8\delta}{n^{1-\delta}} \right\}.
\]

Similarly to Lemma 3, we get instead of (29),

\[
\frac{\tilde{c}''}{2} E_{x,x'} e^{n\delta} 1(B_n) \leq E_{x,x'} e^{(n-1)\delta} \left( e^{k|X_n|^\alpha} + e^{k|X'_n|^\alpha} \right) 1(B_n) \\
- E_{x,x'} e^{n\delta} \left( e^{k|X_{n+1}|^\alpha} + e^{k|X'_{n+1}|^\alpha} \right) 1(B_n) \\
+ 8\delta E_{x,x'} e^{n\delta} e^{k(|X_n| \vee |X'_n|)^\alpha} 1(B_n) 1(\tilde{D}^c).
\]

Now we are able to repeat remaining lines from the proof of Lemma 3, using \(|X_n| \vee |X'_n|\) instead of \(|X_n|\) and other obvious modifications. Finally, we apply Lemma 4 instead of Lemma 2 together with the inequality \(e^{k(|X_n| \vee |X'_n|)^\alpha} \leq e^{k|X_n|^\alpha} + e^{k|X'_n|^\alpha}\). \(\square\)

**Remark 9** Analyzing the proofs of Lemmas 2–5, we see that all the lines can be rewritten in a more general case. For example, let \(\tau_0\) and \(\tau\) be two stopping-times, \(0 \leq \tau_0 < \tau\) and \(\{\tau_0 \leq n < \tau\} \subset \{|X_n| > R\}\) for every \(n\). Then the conclusion of Lemma 2 would be

\[
E_x e^{k|X_n|^\alpha} 1(\tau_0 \leq n < \tau) \leq E_x e^{k|X_{\tau_0}|^\alpha} 1(\tau_0 \leq n).
\]

Moreover, if \(\mathcal{L}(X_0) = \mu\), then one may write a similar inequality with \(E_\mu\) instead of \(E_x\), having integrated the last inequality with respect to the measure \(\mu\). Integration is possible once \(\mu\) has finite sub-exponential moments of the same order as in (5).

**PROOF of Lemma 6.** Consider the process on \(B = \{x \in \mathbb{R}^d : |x| \leq R\}\). Because of the local Doeblin condition the process has an invariant measure \(\mu^B_\infty\). Denote \(\tau = \inf\{n \geq 1 : X_n \in B\}\). Due to the Harris representation, the invariant measure \(\mu_\infty\) for \((X_n)\) is equal to

\[
\mu_\infty(A) = \frac{1}{c(B)} \int_B \mu^B_\infty(dx) E_x \sum_{n=1}^{\tau} 1(X_n \in A),
\]

where \(c(B)\) is the capacity of the set \(B\).
where \( c(B) = \int_B \mu_{\infty}^B (dx) \mathbb{E}_x \tau \) (see Proposition 10.4.8 and Theorem 10.4.9 in [34]). Thus, for any non-negative function \( h \),

\[
\int_{\mathbb{R}^d} h(x) \mu_{\infty}(dx) = \frac{1}{c(B)} \int_B \mu_{\infty}^B (dx) \mathbb{E}_x \sum_{n=1}^{\tau} h(X_n).
\]

In our case \( h(x) = e^{k|x|^\alpha} \) and

\[
\mathbb{E}_x \sum_{n=1}^{\tau} h(X_n) \leq \sum_{n=1}^{\tau-1} \mathbb{E}_x e^{k|X_n|^\alpha} + e^{kR^\alpha}.
\]

If \( \tau > 1 \), then \( \{1 \leq n < \tau\} \subset \{|X_n| > R\} \). Hence, keeping in mind Remark 9, we may apply Lemma 2 with \( k + \varepsilon \) instead of \( k \), where \( \varepsilon < (K-k)/2 \).

If \( R \) is large enough, then \( e^{k|x|^\alpha} \leq e^{(k+\varepsilon)|x|^\alpha} |x|^{-1-p+\alpha} \) for \( |x| > R \), so we deduce from (12),

\[
\sum_{n=1}^{\tau-1} \mathbb{E}_x e^{k|X_n|^\alpha} \leq \sum_{n=1}^{\tau-1} \mathbb{E}_x e^{(k+\varepsilon)|X_n|^\alpha} |X_n|^{-1-p+\alpha} \leq \frac{1}{p'} \mathbb{E}_x e^{(k+\varepsilon)|X_1|^\alpha}.
\]

The last term is finite. Hence, the desired integral over \( B \) is finite, too. \( \square \)

**Remark 10** In the following proofs we will refer to the coupling method. See [35,43–45] for details.

Consider the pair of independent copies of the Markov process \((X_n, X'_n)\) with initial data \( \mathcal{L}(X_0) = \mu \) and \( \mathcal{L}(X'_0) = \mu' \). Let \( m \) be an arbitrary non-negative integer, and introduce the sequence of stopping-times \( \gamma_1 < \gamma_2 < \ldots \) (they also depend on the index \( m \) which we drop) by formulas

\[
\gamma_1 = \inf \{ n \geq m : |X_n| \vee |X'_n| \leq \tilde{R} \},
\]

\[
\gamma_{j+1} = \inf \{ n > \gamma_j : |X_n| \vee |X'_n| \leq \tilde{R} \},
\]

where \( \tilde{R} \) is chosen in Lemma 4.

If the process \( X_n \vee X'_n = X_n 1(|X_n| \geq |X'_n|) + X'_n 1(|X_n| < |X'_n|) \) is positive recurrent with respect to the ball \( \{ x : |x| \leq \tilde{R} \} \), then we can extend the probability space (without changing notations for probability and expectation) and define a new process \((\tilde{X}_n)\) which is equivalent in distribution to \((X_n)\), and and a stopping-time \( L_m \) with respect to \( \tilde{\mathcal{F}}_n = \mathcal{F}^X_{\leq n},X',\tilde{X} \), such that

\[
\mathbb{P}(\tilde{X}_n = X_n, n \leq L_m) = \mathbb{P}(\tilde{X}_n = X'_n, n \geq L_m) = 1.
\]
Due to the coupling inequality one has,
\[ \text{var}_{B \in \mathcal{F}_{x \geq n+m}} (\mathbb{P}(B|\mathcal{F}_m) - \mathbb{P}(B)) \leq \mathbb{P}(L_m > n + m|\tilde{\mathcal{F}}_m). \] (31)

(Cf. to (30) regarding the question why in (31) $\sigma$-fields in both sides look different; in fact, they contain the same information and, hence, are equal.) The local Doeblin condition and the Markov property imply the inequality
\[ \sup_m \mathbb{P}(L_m > \gamma_j|\tilde{\mathcal{F}}_m) \leq (1 - \kappa)^{j-1}, \quad j \geq 1, \] (32)

with $\kappa = \kappa(\tilde{R})$. Remind that $\kappa(\tilde{R}) \in (0, 1]$. Along with inequalities on stopping times $\gamma_j$, the bounds (30) and (31) provide us an easy way to estimate $\beta_{n,x}$ which will be used in the sequel.

**PROOF of the inequality (9) in Theorem 1** is similar to the proof of Theorem 1 in [44]. We will use the coupling method with $\mu$ concentrated in $x \in \mathbb{R}^d$, $\mu' = \mu_\infty$, and $m = 0$ accordingly to Remark 10 to establish (9).

By Lemmas 5 and 6 and Remark 9,
\[ \mathbb{E}_{\mu,\mu'} e^{\gamma_j} \leq C(x) + C(\mu_\infty) < \infty, \quad C(x) = C' e^{k|x|^\alpha}, \quad k < K. \]

Further,
\[ \mathbb{E}_{\mu,\mu'} e^{(\gamma_{j+1} - \gamma_j)} \leq e \mathbb{E}_{\mu,\mu'} 1(\gamma_{j+1} = \gamma_j + 1) + e \mathbb{E}_{\mu,\mu'} e^{(\gamma_{j+1} - (\gamma_j + 1))} 1(\gamma_{j+1} > \gamma_j + 1). \]

Because of $|X_{\gamma_j}| \lor |X'_{\gamma_j}| \leq \tilde{R}$ and (4)–(5), we have at the next step,
\[ \mathbb{E}_{\mu,\mu'} \left( e^{k|X_{\gamma_{j+1}}|^\alpha} + e^{k|X'_{\gamma_{j+1}}|^\alpha} \right) \leq M < \infty. \]

On the set $\{ \gamma_{j+1} > \gamma_j + 1 \}$ we may apply Lemma 5 and Remark 9 again. Thus, one can choose $C > 0$ such that
\[ \mathbb{E}_{\mu,\mu'} e^{(\gamma_{j+1} - \gamma_j)} \leq C. \]

By the strong Markov property,
\[ \mathbb{E}_{\mu,\mu'} e^{\gamma_{j+1}} \leq \mathbb{E}_{\mu,\mu'} e^{(\gamma_{j+1} - \gamma_j)} e^{(\gamma_j - \gamma_{j-1})} \ldots e^{\gamma_1} \leq (C(x) + C(\mu_\infty)) C^j, \]
and due to Bienaimé-Chebyshev’s inequality
\[
P_{\mu,\mu'}(\gamma_{j+1} > n) \leq e^{-n^\delta} \mathbb{E}_{\mu,\mu'} e^{\gamma_{j+1}^\delta} \leq (C(x) + C(\mu_\infty))C^j e^{-n^\delta}.
\] (33)

Hence, Hölder’s inequality with \(1/a + 1/b = 1\), \(a > 1\), \(b > 1\), (33) and (32) imply
\[
\text{var}(\mu_n - \mu_\infty) \leq \mathbb{P}_{x,\mu_\infty}(L_0 > n) \\
\leq \sum_{j=1}^{\infty} \mathbb{E}_{x,\mu_\infty} e^{k|X_n|^\alpha} 1(\gamma_i \leq n < \gamma_{i+1}) \\
\leq \sum_{j=1}^{\infty} \mathbb{P}_{x,\mu_\infty}^{1/a}(L_0 > \gamma_j) \mathbb{P}_{x,\mu_\infty}^{1/b}(\gamma_{j+1} > n) \\
\leq C(x, \mu_\infty) e^{-n^\delta/b} \sum_{j=1}^{\infty} (1 - \kappa) j/a C^{j/b},
\]
with \(C(x, \mu_\infty) \leq C e^{k|x|^\alpha}, k < K\).

Let us fix a large \(b\) providing \((1 - \kappa)^{1/a} C^{1/b} < 1\). Remind that here \(\kappa = \kappa(\tilde{R})\) with \(\tilde{R}\) chosen in Lemma 4. Then the series converges, and one gets a bound
\[
\text{var}(\mu_n - \mu_\infty) \leq C(x, \mu_\infty, \kappa, b) e^{-n^\delta/b}, \quad \text{with some } k < K \text{ and } C(x, \mu_\infty, \kappa, b) \leq C(\kappa, b) e^{k|x|^\alpha}.
\]

PROOF of Lemma 7. We apply the coupling method similarly to the first part of the proof of Theorem 1. One writes,
\[
\mathbb{E}_{x} e^{k|X_n|^\alpha} \leq \mathbb{E}_{x,\mu_\infty} e^{k|X_n|^\alpha} - e^{k|X'_n|^\alpha} + \mathbb{E}_{\mu_\infty} e^{k|X'_n|^\alpha}.
\]

The first term in the right-hand side which is obviously equal to
\[
\mathbb{E}_{x,\mu_\infty} e^{k|X_n|^\alpha} - e^{k|X'_n|^\alpha} 1(n < L_0) \leq \mathbb{E}_{x,\mu_\infty} e^{k|X_n|^\alpha} 1(n < L_0) + \mathbb{E}_{\mu_\infty} e^{k|X'_n|^\alpha}.
\]

Take \(a > 1\) to provide \(ka < K\). By virtue of Lemma 4, there is a constant \(M = M(x, \mu_\infty)\) such that
\[
\mathbb{E}_{x,\mu_\infty} e^{ka|X_n|^\alpha} 1(\gamma_i \leq n < \gamma_{i+1}) \leq M.
\]

Applying Hölder’s inequality with \(1/a + 1/b = 1\), \(a > 1\), \(b > 1\), one gets,
\[
\mathbb{E}_{x,\mu_\infty} e^{k|X_n|^\alpha} 1(n < L_0) \leq \sum_{i=1}^{\infty} \mathbb{E}_{x,\mu_\infty} e^{k|X_n|^\alpha} 1(\gamma_i \leq n < \gamma_{i+1}) 1(\gamma_i < L_0)
\]

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\[
\leq \sum_{i=1}^{\infty} \mathbb{P}_{x,\mu,\infty}^{1/a} e^{ka|X_n|^{\alpha}} 1(\gamma_i \leq n < \gamma_{i+1}) \mathbb{P}_{x,\mu,\infty}^{1/b} (L_0 > \gamma_i)
\]
\[
\leq \sum_{i=1}^{\infty} M_{1/a} (1 - \kappa)^{(i-1)/b} < \infty.
\]

Since \(X'_n\) is distributed according to the invariant measure, \(\mathbb{E}_{\mu,\infty} e^{k|X'_n|^{\alpha}}\) is finite due to Lemma 6. □

**PROOF of inequality (10) in Theorem 1.** To prove the rate of \(\beta\)-mixing, we repeat the first part of the proof with an arbitrary \(m \geq 0\). The only new feature is the estimate for \(\sup_m \mathbb{E}_{\mu,\mu'} e^{(\gamma_1 - m)^h}\) (remind that \(\gamma_1\), in fact, depends on \(m\)). Lemma 7 along with Lemma 5 show that this value is finite. Due to (31) we get

\[
\beta_{n,x} \leq \sup_m \mathbb{P}(L_m > n + m|\mathcal{F}_m).
\]

The latter probability is then estimated using (32), similarly to the proof of the bound (8). Thus, (10) holds true. □

**Remark 11** Detailed inspection of the proofs in this article shows us that the results of [44] is valid under weaker restrictions on the sequence of noise. Namely, instead of i.i.d. random variables one can take a sequence satisfying (4) with finite moments of the order required in lemmas and theorems in [44].

**References**


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